Math 31 - Homework 3

Due Friday, July 13

Easy

1. Let G be a group of order pq, where p and q are prime numbers. Show that every *proper* subgroup of G is cyclic.

2. We proved in class that every subgroup of a cyclic group is cyclic. The following statement is almost the converse of this:

"Let G be a group. If every *proper* subgroup of G is cyclic, then G is cyclic."

Find a counterexample to the above statement.

3. [Herstein, Section 2.4 #1] Verify that the relation $\sim$ is an equivalence relation on the set S given.

(a) $S = \mathbb{R}$, and $a \sim b$ if $a - b$ is rational.

(b) $S = \mathbb{C}$, and $a \sim b$ if $|a| = |b|$.

(c) $S = \{\text{straight lines in the plane}\}$, and $a \sim b$ if a, b are parallel.

(d) $S = \{\text{all people}\}$, and $a \sim b$ if they have the same color eyes.

4. [Herstein, Section 2.4 #2] The relation $\sim$ on the real numbers $\mathbb{R}$ defined by $a \sim b$ if both $a > b$ and $b > a$ is *not* an equivalence relation. Why not? What properties of an equivalence relation does it satisfy?

Medium

5. Let r and s be positive integers, and define

$$H = \{nr + ms : n, m \in \mathbb{Z}\}.$$

(a) Show that H is a subgroup of $\mathbb{Z}$.

(b) We saw in class that every subgroup of $\mathbb{Z}$ is cyclic. Therefore, $H = \langle d \rangle$ for some $d \in \mathbb{Z}$. What is this integer d? Prove that the d you've found is in fact a generator for H.

6. Let a and b be elements of a group G. Show that if ab has finite order n, then ba also has order n.

7. Let H be a subgroup of a group G and let $g \in G$. Define a one-to-one map of H onto Hg. Prove that your map is one-to-one and onto.

8. We will see in class that if p is a prime number, then the cyclic group $\mathbb{Z}_p$ has no proper subgroups as a consequence of Lagrange's theorem. This problem will have you investigate a "converse" to this result.

(a) [Herstein, Section 2.3 #14] If G is a group which has no proper subgroups, prove that G must be cyclic.

(b) [Herstein, Section 2.3 #15] Extend the result of (a) by showing that if G has no proper subgroups, then G is not only cyclic, but

$$|G| = p$$

for some prime number p.

Hard

9. Let $G = \langle a \rangle$ be a cyclic group of order n. Prove that for any positive divisor m of n, G has exactly one subgroup of order m. [Hint: You may want to use the formula that relates $\left|a^j\right|$ to $|a|$.]

10. [Herstein, Section 2.4 #8] Let G be a group with $H \leq G$, and for $a \in G$ define

$$aHa^{-1} = \left\{aha^{-1} : h \in H\right\}.$$

If every right coset of H in G is a left coset of H in G, prove that $aHa^{-1} = H$ for all $a \in G$. [**Note:** To say that a left coset aH is also a right coset does not necessarily mean that $aH = Ha$. It only means that $aH = Hb$ for some $b \in G$. However, you will be able to show that $Hb = Ha$ in this case.]